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# INSTABILITY OF A SELF-GRAVITATING COMPRESSIBLE MEDIUM

**Yu. G. Gubarev** UDC 532.2

Widely known in analytical mechanics are the theorems of Lyapunov and Chetaev (inversions of the Lagrange theorem) [1, 2], consisting in proof of the instability of the equilibrium position of a mechanical system with the absence in the system of a minimum of the potential energy. A way to generalize these theorems to systems containing rigid bodies and a fluid was proposed by Rumyantsev [3, 4]. This approach was further developed in [5-10], where the instability of several fluid equilibria was proved with the aid of the Lyapunov virial functional [3, 11].

In the present work we consider an example of inversion of the Lagrange theorem in the hydrodynamics of a self-gravitating fluid. We study the problem of the stability of the states of equilibrium (rest) of an infinite self-gravitating compressible medium [12-14]. It is proved by the Lyapunov direct method that the system is unstable if there exist small disturbances of the density that reduce the potential energy. Two-sided estimates of the growth of the disturbances are obtained in the linear approximation. The lower estimate guarantees exponential increase of the energy of the gravitational field. The upper estimate shows that the disturbances grow no faster than exponentially. In both cases the exponents are calculated on the basis of the parameters of the states of equilibrium and the initial values for the disturbance fields. On the basis of the relations of the exact problem there is obtained a lower estimate, indicating the rms growth of the disturbances of the density and/or the velocity potential.

We note that from the mathematical viewpoint these estimates have an *a priori* nature, since the corresponding solution existence theorems have not been proved.

**1. Introduction.** The idea that the observable structuring of matter in the Universe is due to gravitational interaction was expressed by Newton [15]. However, this hypothesis was first given a modern mathematical formulation only by Jeans [12, 13], who examined the linear problem of the stability of the states of rest of an infinite self-gravitating compressible medium with undisturbed density $\rho_\infty$ = const. He showed that small disturbances of the density

$$\rho' = b\exp[i(\mathbf{k}\mathbf{x} - \omega t)],\ b = \text{const},$$

will grow with time if the wavenumbers **k** satisfy the condition

$$|\mathbf{k}| < k_*,\ k_* \equiv 4\pi G\rho_\infty c^{-2}. \tag{1.1}$$

Here $\mathbf{x} = (x_1, x_2, x_3)$ are the Cartesian coordinates; $\omega$ is the wave frequency; t is the time; c is the speed of sound; G is the gravitational constant.

In later years studies were conducted basically in two directions: 1) analysis of the influence of various factors (rotation, magnetic field, turbulence) on the Jeans criterion (1.1) [16-19]; 2) obtaining the conditions of stability of the states of rest of an infinite self-gravitating compressible medium relative to one-dimensional disturbances of finite amplitude [20-24]. A characteristic feature of the studies [16-24] is that in them the conclusions relating to instability are drawn simply on the basis of the presence of disturbances that decrease the energy of the states of rest; the authors thereby accept the validity of the inverse Lagrange theorem without any foundation for this.

The primary objective of the present work is to prove the instability of the states of equilibrium (rest) of an infinite self-gravitating compressible medium if the Lagrange theorem inversion condition is satisfied. This condition means the absence

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in these states of equilibrium of a minimum of the potential energy.

**2. Problem Statement**. The three-dimensional adiabatic motions of a self-gravitating ideal compressible fluid are studied. We use the notations: p, $\rho$, and $v = (v_1, v_2, v_3)$ are the pressure, density, and velocity fields, $\Phi = \Phi(\mathbf{x}, t)$ is the gravitational field potential, $\gamma$ is the adiabatic exponent. It is assumed that the fluid fills the entire space, and at infinity (i.e., for $|\mathbf{x}| \to \infty$) it has constant density and is at rest [14]. The motions of the fluid are described by the solutions of the system of equations [14, 21]

$$\rho D\mathbf{v} = -\nabla p - \rho\nabla\Phi,\ D\rho + \rho\,\mathrm{div}\mathbf{v} = 0,\ \Delta\Phi = 4\pi G(\rho - \rho_\infty), \quad D \equiv \partial/\partial t + \mathbf{v}\nabla, \tag{2.1}$$

supplemented by the thermodynamic relation

$$p = a\rho^\gamma,\ a = \mathrm{const},\ \gamma = c_p/c_V > 1, \tag{2.2}$$

and by the conditions at infinity

$$|\mathbf{v}| \to 0,\ |\nabla\Phi| \to 0, \rho \to \rho_\infty, p \to p_\infty \quad \text{for} \quad |\mathbf{x}| \to \infty, \tag{2.3}$$

where $c_p$ and $c_v$ are the specific heats: $\rho_\infty$, $p_\infty$ are constant quantities. The initial values for (2.1)-(2.3) are specified in the form

$$\rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}),\ \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}) \tag{2.4}$$

with obvious limitations on the functions $\rho_0(\mathbf{x})$ and $v_0(\mathbf{x})$. All the utilized functions and their derivatives, entering into the equations of motion, the thermodynamic relation, and the conditions at infinity, are considered to be continuous.

For the solutions of the problem (2.1)-(2.4) there exists the energy integral (here the hereafter integration is performed over the entire space)

$$dE_1/dt = 0,\ E_1 = K_1 + \Pi_1 = \mathrm{const},\ 2K_1 = \int \rho v_i v_i d\mathbf{x},\ d\mathbf{x} \equiv dx_1 dx_2 dx_3, \quad \Pi_1 = \int (p(\gamma - 1)^{-1} + (\rho - \rho_\infty)\Phi/2)\, d\mathbf{x}. \tag{2.5}$$

The states of hydrostatic equilibrium are solutions of the problem (2.1)-(2.4) of the form

$$\mathbf{v} = \mathbf{v}_0(\mathbf{x}) \equiv 0, \rho = \rho_0(\mathbf{x}) = \rho_\infty, p = p_0(\mathbf{x}) = p_\infty,\ \Phi = \Phi_0(\mathbf{x}) = \Phi_\infty, \tag{2.6}$$

where $\Phi_\infty$ is a constant quantity.

**3. Extremum condition**. Let $\delta v_i$, $\delta\rho$, $\delta p$, and $\delta\Phi$ be the variations of the fields $v_i$, $\rho$, p, and $\Phi$, respectively. With the aid of Eq. (2.1) and the relation (2.2) we find the connections of $\delta\rho$ with $\delta\Phi$ and $\delta p$:

$$\Delta\delta\Phi = 4\pi G\delta\rho,\ \delta p = \gamma p_\infty \rho_\infty^{-1}\delta\rho. \tag{3.1}$$

For the first variation $\delta E_i$, calculated from the solution (2.6) with account for (3.1), the following representation is valid

$$\delta E_1 = \int [(\gamma p_\infty/(\gamma - 1)\rho_\infty + \Phi_\infty/2)\delta\rho]\, d\mathbf{x}. \tag{3.2}$$

Using the arbitrariness of the quantity $\Phi_\infty$, we can select it so as to satisfy the relation

$$\Phi_\infty = -2\gamma p_\infty ((\gamma - 1)\rho_\infty)^{-1}. \tag{3.3}$$

After this we obtain $\delta E_i = 0$. It is clear that the equality (3.3) agrees with (2.6) and with (2.1).

Thus, if we take as $\Phi_\infty$ in (3.2) the quantity satisfying the condition (3.3), then the states of rest (2.6) are stationary points of the functional $E_i$ (2.5) in the class of the independent variations $\delta v_i$, $\delta\rho$, subject only to the conditions of continuity together with their first derivatives.

**4. Second Variation.** The expression for the second variation of the energy $E_i$ (2.5) has the form

$$\delta^2 E_1 = \int [(\rho_\infty/2)\delta v_i \delta v_i + (\gamma/2)p_\infty \rho_\infty^{-2}(\delta\rho)^2 + (\delta\rho/2)\delta\Phi]\, dx. \tag{4.1}$$

A remarkable property of $\delta^2 E_i$ (4.1) is its conservation by virtue of the linearization on the exact stationary solution (2.6) of the problem (2.1)-(2.4). This property follows from the fact that $E_i$ (2.5) is the energy integral for the exact problem (2.1)-(2.4), and the first variation $\delta E_1$ (3.2) vanishes on (2.6).

**5. Linearized Problem.** Actually, linearization of the relations (2.1)-(2.3) on the solution (2.6) yields

$$\begin{gathered}\rho_\infty \partial v_i'/\partial t = -\partial p'/\partial x_i - \rho_\infty \partial\Phi'/\partial x_i,\ \partial\rho'/\partial t = -\rho_\infty \partial v_i'/\partial x_i, \\ \partial^2\Phi'/\partial x_i^2 = 4\pi G\rho',\ p' = \gamma p_\infty \rho'/\rho_\infty,\ |\mathbf{x}| \to \infty: p', \rho', |\nabla\Phi'|, |\mathbf{v}'| \to 0.\end{gathered} \tag{5.1}$$

Here $\mathbf{v}'$, $\rho'$, $p'$, $\Phi$ are the fields of the disturbances of the velocity, density, pressure, and the gravitational field potential. The initial values (2.4) in the linear approximation reduce to the form

$$\rho'(\mathbf{x}, 0) = \rho_0'(\mathbf{x}),\ \mathbf{v}'(\mathbf{x}, 0) = \mathbf{v}_0'(\mathbf{x}). \tag{5.2}$$

In the following the primes on the disturbance fields, distinguishing them from the complete solutions of the system of Eq. (2.1), are dropped.

Direct calculations show that for the solutions of the problem (5.1), (5.2) the analog of the energy integral is valid

$$\begin{gathered}dE/dt = 0,\ E = K + \Pi = \text{const}, \\ 2K = \int \rho_\infty v_i v_i d\mathbf{x},\ 2\Pi = \int \rho\, (\Phi + \gamma p_\infty \rho_\infty^{-2}\rho)\ d\mathbf{x}.\end{gathered} \tag{5.3}$$

Comparison of the functional E (5.3) with the second variation $\delta^2 E_1$ (4.1) actually discloses their coincidence if the variations $\delta v_i$, $\delta\rho$, $\delta_p$, and $\delta\Phi$ are interpreted as infinitesimal Eulerian disturbances of the corresponding hydrodynamic fields.

The further examination is limited to that class of motions for which the disturbances involve only displacements of the fluid particles from the equilibrium position [6]. This class of motions is most simply described with the aid of the Lagrangian displacement field $\xi = \xi(\mathbf{x}, t)$ [11]:

$$\partial\xi_i/\partial t = v_i. \tag{5.4}$$

In this case the relations (5.5) are rewritten as follows:

$$\begin{gathered}\rho_\infty \partial^2\xi_i/\partial t^2 = -\partial p/\partial x_i - \rho_\infty \partial\Phi/\partial x_i,\ \rho = -\rho_\infty \partial\xi_i/\partial x_i, \\ p = -\gamma p_\infty \partial\xi_i/\partial x_i,\ \partial^2\Phi/\partial x_i^2 = -4\pi G\rho_\infty \partial\xi_i/\partial x_i,\ |\mathbf{x}| \to \infty: |\nabla\Phi|, |\xi| \to 0.\end{gathered} \tag{5.5}$$

The initial values (5.2) for (5.4), (5.5) take the form

$$\xi(\mathbf{x}, 0) = \xi_0(\mathbf{x}),\ \mathbf{v}(\mathbf{x}, 0) = \partial\xi(\mathbf{x}, 0)/\partial t = \mathbf{v}_0(\mathbf{x}). \tag{5.6}$$

It is important to emphasize that in this and the subsequent sections it is assumed that the utilized integrals of the fields of the disturbances of the velocity, density, and the gravitational field potential, and also the integrals of the Lagrangian displacement fields, exist and are finite. This assumption is based on the hypothesis relating to the continuity of the permissible disturbances and on the suitable conditions of their decay at infinity.

In the following we shall prove the instability of the states of equilibrium (2.6) in the linear approximation with the absence in them of a minimum of the potential energy II (5.3), we shall derive the estimates of the growth of the disturbances and formulate an example illustrating the obtained results.

**6. Inversion of the Lagrange Theorem and the Lyapunov Functional.** In terms of the Lagrangian displacements $\xi$ the condition of inversion of the Lagrange theorem (i.e., the absence of a minimum of the potential energy $\Pi$ (5.3) on the solution (2.6)) means that there exists that set Q of the initial fields $\xi_0(\mathbf{x})$ (5.6) for which

$$\Pi = \Pi_* < 0 \text{ for } \xi_0(\mathbf{x}) \in Q. \tag{6.1}$$

If however, $\xi_0(\mathbf{x}) \notin Q$, then the inequality (6.1) can be replaced by the opposite inequality, i.e., the states of rest (2.6) are the infinite-dimensional analogs of the "saddle" point of the functional II. The meaning of the requirement (6.1) can be clarified by writing the functional II (5.3) with the aid of the conditions at infinity for the problem (5.1), (5.2) in the form of a sum

$$\Pi = A_1 + A_2,\ A_1 \equiv -(8\pi G)^{-1} \int (\partial\Phi/\partial x_i)^2\, d\mathbf{x},$$
$$A_2 \equiv \int ((\gamma/2) p_\infty \rho_\infty^{-2} \rho^2)\, d\mathbf{x}.$$

We see that the condition (6.1) is satisfied if and only if among the initial Lagrangian displacement fields $\xi_0(\mathbf{x})$ there are those for which the inequality $A_2 < |A_1|$ will be valid.

For the demonstration of the instability there are introduced the functionals [6-10]

$$M \equiv \int \rho_\infty \xi_i \xi_i d\mathbf{x},\ M^{\cdot}/2 = W \equiv \int \rho_\infty \xi_i v_i d\mathbf{x}. \tag{6.2}$$

Differentiation of the doubled functional $W$ with respect to time and subsequent transformations with the use of (5.1), (5.3), (5.5), (6.2) yield the relation

$$M^{\cdot\cdot} = 2W^{\cdot} = 4(K - \Pi) = 8K - 4E, \tag{6.3}$$

which is termed the virial equality [11]. The relation (6.3) is multiplied by the undetermined constant multiplier $\lambda$ and is combined with (5.3). After simple transformations, the obtained relation reduces to the form [8, 9]

$$E_\lambda^{\cdot} = 2\lambda E_\lambda - 4\lambda K_\lambda,\ E_\lambda = K_\lambda + \Pi_\lambda,\ 2\Pi_\lambda = 2\Pi + \lambda^2 M,$$
$$2K_\lambda = 2K - \lambda M^{\cdot} + \lambda^2 M = \int \rho_\infty (\partial\xi/\partial t - \lambda\xi)^2 d\mathbf{x}. \tag{6.4}$$

The limitation $\lambda > 0$ is imposed. After this there follows from (6.4) by virtue of the non-negativity of the functional $K_7$ the inequality $E_\lambda^{\cdot} \le 2\lambda E_\lambda$, integration of which makes it possible to obtain the relation

$$E_\lambda(t) \leqslant E_\lambda(0)\exp(2\lambda t). \tag{6.5}$$

It is important to emphasize that the inequality (6.5) is valid for any solutions of the problem (5.4)-(5.6) and for any positive value of the parameter $\lambda$. Moreover, in the derivation of (6.5) no limitations on the sign of the functional $\Pi$ are required.

The inequality (6.5) shows that the quantity $E_\lambda$ varies monotonically with time. This makes it possible to use it in the following as the Lyapunov functional [1-3].

**7. Lower Estimate of the Growth of the Disturbances.** Let the condition (6.1) be valid. This means that we can take as the initial Lagrangian displacement fields those functions $\xi_0(\mathbf{x})$ (5.6) for which the inequality $\Pi(0) < 0$ holds; as the initial velocity fields we examine the functions $\mathbf{v}_0(\mathbf{x})$ (5.6) such that the relation $K(0) < |\Pi(0)|$ is satisfied. Then the following inequality is valid

$$E(0) < 0. \tag{7.1}$$

In accordance with the definition (6.4), the functional $E_\lambda(0)$ is a polynomial of second degree of $\lambda$ with a positive coefficient M(0) (6.2) with $\lambda^2$ and a negative free term E(0) (7.1):

$$E_\lambda(0) = E(0) - (\lambda/2)M'(0) + \lambda^2 M(0). \quad (7.2)$$

Setting $\lambda > 0$, with the aid of (7.2) we can show that on the interval

$$0 < \lambda < \Lambda_1 = B + C^{1/2},\ B = \frac{\lambda M'(0)}{4M(0)},\ C = B^2 - \frac{E(0)}{M(0)} \quad (7.3)$$

the relation is satisfied

$$E_\lambda(0) < 0. \quad (7.4)$$

It follows from the inequalities (6.5), (7.4) that in the course of time the solutions of the problem (5.4)-(5.6) grow exponentially.

If $\lambda = \lambda_1 - \delta$ (with any $\delta$ from the interval $0 < \delta < \Lambda_1$), then the relation (6.5) takes the form

$$E_{\Lambda_1-\delta}(t) \leqslant E_{\Lambda_1-\delta}(0)\exp\left[2(\Lambda_1 - \delta)t\right] (E_{\Lambda_1-\delta}(0) < 0). \quad (7.5)$$

By virtue of the definition of the functionals $K_\lambda$ and $\Pi_\lambda$ (6.4), the inequality holds

$$E_\lambda(t) = K_\lambda(t) + \Pi_\lambda(t) > -(8\pi G)^{-1}\int (\partial\Phi/\partial x_i)^2 d\mathbf{x},$$

which together with (7.5) yields the estimate

$$\int (\partial\Phi/\partial x_i)^2 d\mathbf{x} > 8\pi G|E_{\Lambda_1-\delta}(0)|\exp\left[2(\Lambda_1 - \delta)t\right]. \quad (7.6)$$

It follows from (7.6) that the parameter $\Lambda_1$ (7.3) is the lower estimate of the increments of the solutions of the problem (5.4)-(5.6).

The estimate (7.6) can be improved if we examine that class of the solutions of the problem (5.4)-(5.6) for which the initial velocity field $\mathbf{v}_0(\mathbf{x})$ and Lagrangian displacement field $\xi_0(\mathbf{x})$ are connected at each point by the relations

$$\mathbf{v}_0(\mathbf{x}) = \lambda\xi_0(\mathbf{x}). \quad (7.7)$$

In fact, in this case there follows from (6.4), (7.7)

$$K_\lambda(0) = 0,\ E_\lambda(0) = \Pi_\lambda(0). \quad (7.8)$$

If $\lambda > 0$ and the condition (6.1) is satisfied for the Lagrangian displacement fields $\xi_0(\mathbf{x})$ (5.6), then, since by virtue of the third relation (6.4)

$$2\Pi_\lambda(0) = 2\Pi(0) + \lambda^2 M(0),$$

on the interval

$$0 < \lambda < \Lambda = \left(-\frac{2\Pi(0)}{M(0)}\right)^{1/2} \quad (7.9)$$

there holds the inequality $\Pi_\lambda(0) < 0$. Setting $\lambda = \Lambda - \delta_1$ (with arbitrary $\delta_1$ from the interval $0 < \delta_1 < \Lambda$) and considering (7.8), we can write the inequality (6.5) in the form

$$E_{\Lambda-\delta_1}(t) \leq \Pi_{\Lambda-\delta_1}(0)\exp[2(\Lambda - \delta_1)t],$$

from which there follows the estimate

$$\int (\partial\Phi/\partial x_i)^2 dx > 8\pi G|\Pi_{\Lambda-\delta_1}(0)|\exp[2(\Lambda - \delta_1)t]. \quad (7.10)$$

Thus, the parameter $\Lambda$ (7.9) is the lower estimate of the increments of the solutions of the problem (5.4)-(5.6) from the class (7.7).

Comparison of the estimates (7.6), (7.10) indicates that the solutions of the problem (5.4)-(5.6) from the class (7.7) grow faster than the solutions from the class (7.1).

It is shown below that the disturbances (7.7) are most dangerous, since the fastest growth of the solutions of the problem (5.4)-(5.6) is observed for

$$\Lambda^+ = \sup_{\xi_0(x) \in \Omega}\Lambda. \quad (7.11)$$

**8. Upper Estimate of the Growth of the Disturbances.** Let there hold the inequality $\lambda > \Lambda^+$ (7.11). Then for the initial Lagrangian displacement fields $\xi_0(\mathbf{x})$ (5.6), (6.1) the relation holds

$$\Pi_\lambda(0) > 0. \quad (8.1)$$

The inequality (8.1) is even more satisfied for those Lagrangian displacement fields $\xi_0(\mathbf{x})$ which do not satisfy the condition (6.1). Thus, the functional $\Pi_\lambda$ is positive definite for all possible initial Lagrangian displacement fields $\xi_0(\mathbf{x})$ (5.6). The relations (5.1), (5.4), (6.4) show that the functional $E_\lambda$ is also positive definite for all possible initial Lagrangian displacement fields $\xi_0(\mathbf{x})$ and velocity fields $\mathbf{v}_0(\mathbf{x})$ (5.6).

Consequently, for $\lambda = \Lambda^+ + \varepsilon_1$ ($\varepsilon_1 > 0$) there follows from the basic inequality (6.5) the estimate

$$E_{\Lambda^+ + \varepsilon_1}(t) \leq E_{\Lambda^+ + \varepsilon_1}(0)\exp[2(\Lambda^+ + \varepsilon_1)t]. \quad (8.2)$$

It follows from (8.2) that the parameter $\Lambda^+ + \varepsilon_1$ is the upper estimate of the increments of the solutions of the problem (5.4)-(5.6).

Comparison of the inequalities (7.10) and (8.2) which account for the definition (7.1) makes it possible to conclude that the parameter $\Lambda^+$ provides both the lower and the upper estimates of the rate of growth of the solutions of the problem (5.4)-(5.6) from the class (7.7):

$$\Lambda^+ - \delta_1 \leq \omega_* \leq \Lambda^+ + \varepsilon_1. \quad (8.3)$$

The estimate (8.3) means that those solutions of the problem (5.4)-(5.6) for which the increment is equal to $\Lambda^+$ (7.11) grow most rapidly. Consequently, after calculating the value of $\Lambda^+$ from the formulas (7.9), (7.11) we can answer the question: after what characteristic time will the system "depart" from the given position of equilibrium (2.6)?

**9. Example.** Let the field $\xi(\mathbf{x})$ (5.6) have the form

$$\xi_0 = (\xi_{01}, \xi_{02}, \xi_{03}) = (\xi_{01}(x_1), 0, 0),$$
$$\xi_{01} = \begin{cases} a_1\sin^3\alpha_1 x_1; \ 0 \leq x_1 \leq 2\pi/\alpha_1, \\ 0; \ x_1 \leq 0, \ x_1 \geq 2\pi/\alpha_1, \end{cases} \quad (9.1)$$

where $a_i$ and $\alpha_1$ are constant quantities and $\alpha_1 > 0$. Then with the aid of the relations (5.5), (9.1) it is not difficult to show that the fields of the disturbances of the density $\rho$ and of the gravitational field intensity $\nabla\Phi$ can be written as follows:

$$\rho = \begin{cases} -(3/2)\alpha_1 a_1 \rho_\infty \sin 2\alpha_1 x_1 \sin\alpha_1 x_1;\ 0 \leq x_1 \leq 2\pi/\alpha_1, \\ 0;\ x_1 \leq 0,\ x_1 \geq 2\pi/\alpha_1, \end{cases}$$

$$\nabla\Phi = (\partial\Phi/\partial x_1,\ \partial\Phi/\partial x_2,\ \partial\Phi/\partial x_3) = (\partial\Phi/\partial x_1(x_1),\ 0,\ 0), \tag{9.2}$$

$$\partial\Phi/\partial x_1 = \begin{cases} -4\pi G\rho_\infty a_1 \sin^3\alpha_1 x_1;\ 0 \leq x_1 \leq 2\pi/\alpha_1, \\ 0,\ x_1 \leq 0,\ x_1 \geq 2\pi/\alpha_1. \end{cases}$$

Direct verification makes it possible to ascertain that the relations (9.1), (9.2) satisfy the conditions at infinity (5.1), (5.5). With account for (9.1), (9.2), the functional Π (5.3) has the form

$$\Pi = (9\pi/16\alpha_1)\gamma p_\infty a_1^2(\alpha_1^2 - (20\pi/9)G\rho_\infty c^{-2}),\ c^2 \equiv \gamma p_\infty/\rho_\infty. \tag{9.3}$$

It follows from (99.3) that the functional Π will take negative values if

$$\alpha_1^2 < (20\pi/9)G\rho_\infty c^{-2}. \tag{9.4}$$

Thus, the Lagrange theorem inversion condition (6.1) is satisfied for the field $\xi_0(\mathbf{x})$ (9.1), (9.4). This means that the states of rest (2.6) are unstable. With the aid of (9.1), (9.4) the estimates of the rate of growth of the disturbances (7.6), (7.10), (8.2) are written out in explicit form, and the increment $\Lambda^+$ (7.9), (7.11) of the fastest growing disturbances is calculated. We note that the condition (9.4) agrees with the result obtained earlier [12, 13].

In the following we shall prove the instability of the states of equilibrium (2.6) by virtue of the relations of the exact problem (2.1)-(2.4) with the condition of validity of the inverse theorem, i.e., with the absence in these states of equilibrium of a minimum of the potentials energy $\Pi_1$ (2.5). It is shown that rms growth of the disturbances of the density and/or the velocity potential takes place.

**10. Basic Relations**. Further studies are limited to examining the class of three-dimensional adiabatic potential motions of a self-gravitating ideal compressible fluid $\mathbf{v} = \nabla\varphi$ ($\varphi = \varphi(\mathbf{x}, t)$ is the velocity potential). In this case the exact problem (2.1)-(2.4) can be rewritten as:

$$\begin{gathered} \partial\varphi/\partial t + (\nabla\varphi)^2/2 = -\Phi - \gamma p((\gamma-1)\rho)^{-1} - \beta, \\ \partial\rho/\partial t + \operatorname{div}(\rho\nabla\varphi) = 0,\ \Delta\Phi = 4\pi G(\rho - \rho_\infty), \\ p = a\rho^\gamma,\ a = \text{const},\ \gamma = c_p/c_V > 1, \\ |\nabla\varphi| \to 0,\ |\nabla\Phi| \to 0,\ \rho \to \rho_\infty,\ p \to p_\infty \text{ for } |\mathbf{x}| \to \infty, \\ \varphi(\mathbf{x}, 0) = \varphi_0(\mathbf{x}),\ \rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}). \end{gathered} \tag{10.1}$$

Here $\beta = \beta(t)$ is an arbitrary function of its argument.

The states of hydrostatic equilibrium are solutions of the problem (10.1) of the form ($\varphi_\infty$ is a constant quantity)

$$\begin{gathered} \varphi = \varphi_0(\mathbf{x}) = \varphi_\infty,\ \rho = \rho_0(\mathbf{x}) = \rho_\infty, \\ p = p_0(\mathbf{x}) = p_\infty,\ \Phi = \Phi_0(\mathbf{x}) = \Phi_\infty. \end{gathered} \tag{10.2}$$

The function $\beta = \beta(t)$ is selected so as to satisfy the previously adopted connection (3.3) between the equilibrium state parameters $\Phi_\infty$, $p_\infty$, and $\rho_\infty$:

$$\beta \equiv \gamma p_\infty((\gamma-1)\rho_\infty)^{-1}. \tag{10.3}$$

The possibility of this selection of the function $\beta$ is based on the following considerations. Let the form of the function $\beta$ be fixed at infinity by means of (10.3). Since in accordance with the assumption made above the fluid at infinity has the density $\rho_\infty$ = const and is at rest, then the connection (10.3) will be valid not only for t = 0, but also for t > 0. If we now continue the function $\beta$ in a continuous fashion within the flow region, then we find that (10.3) will be valid in the entire space, both for t = 0 and for any admissible t > 0.

The exact nonstationary solutions of the problem (10.1) are written in the form

$$
\begin{gathered}
\varphi(\mathbf{x}, t) = \varphi_\infty + \varphi'(\mathbf{x}, t), \rho(\mathbf{x}, t) = \rho_\infty + \rho'(\mathbf{x}, t), \\
p(\mathbf{x}, t) = p_\infty + p'(\mathbf{x}, t), \Phi(\mathbf{x}, t) = \Phi_\infty + \Phi'(\mathbf{x}, t);
\end{gathered} \tag{10.4}
$$

where the functions $\varphi'$, $\rho'$, $p'$, $\Phi'$ (10.4), examined as perturbations of the solutions (10.2), satisfy the relations

$$
\begin{gathered}
\partial\rho'/\partial t + \operatorname{div}(\rho_\infty \nabla\varphi' + \rho'\nabla\varphi') = 0, \\
\frac{\partial\varphi'}{\partial t} + \frac{1}{2}\left(\frac{\partial\varphi'}{\partial x_i}\right)^2 = -\Phi' - \frac{\rho'\Phi_\infty}{\rho_\infty + \rho'} - \frac{\gamma p'}{(\gamma - 1)(\rho_\infty + \rho')}, \\
\Delta\Phi' = 4\pi G\rho', p_\infty + p' = a(\rho_\infty + \rho')^\gamma, \\
|\mathbf{x}| \to \infty: |\nabla\varphi'|, |\nabla\Phi'|, \rho', p' \to 0.
\end{gathered} \tag{10.5}
$$

The initial values (10.1) take the form

$$
\varphi'(\mathbf{x}, 0) = \varphi'_0(\mathbf{x}), \rho'(\mathbf{x}, 0) = \rho'_0(\mathbf{x}) \tag{10.6}
$$

In the following the primes on the functions $\varphi'$, $\rho'$, $p'$, $\Phi'$ are dropped. The energy integral is valid for the solutions of the problem (10.5), (10.6).

$$
\begin{gathered}
dE_2/dt = 0, E_2 = K_2 + \Pi_2 = \text{const}, \\
2K_2 = \int (\rho_\infty + \rho)(\nabla\varphi)^2 d\mathbf{x}, \\
\Pi_2 = \int [p(\gamma - 1)^{-1} + (\Phi_\infty + \Phi)\rho/2] d\mathbf{x}.
\end{gathered} \tag{10.7}
$$

**11. Proof of Instability**. The Lyapunov functional is taken in the form

$$
W_1 = \int \rho\varphi d\mathbf{x}. \tag{11.1}
$$

Differentiation of the functional $W_1$ (11.1) with respect to time with the aid of (10.5) and subsequent transformations with the use of (10.7) yield the relation

$$
\begin{gathered}
dW_1/dt = -2E_2 + 3K_2 + X, \\
X = \int \left\{\frac{1}{2}\rho_\infty\left(\frac{\partial\varphi}{\partial x_i}\right)^2 + \frac{2\gamma p_\infty\rho^2}{(\gamma - 1)\rho_\infty(\rho_\infty + \rho)} + \frac{(2 - \gamma)a(\rho_\infty + \rho)^\gamma}{\gamma - 1} + \frac{a\rho_\infty[\gamma(\rho_\infty + \rho)^\gamma - 2\rho_\infty^\gamma]}{(\gamma - 1)(\rho_\infty + \rho)} - \frac{(\gamma + 2)a\rho_\infty^\gamma\rho}{(\gamma - 1)(\rho_\infty + \rho)}\right\} d\mathbf{x}.
\end{gathered} \tag{11.2}
$$

We can show that for $1 < \gamma \le 2$ the inequality holds

$$
X > 0. \tag{11.3}
$$

In fact, if we set $\gamma = 2$, then the functional X (11.3) takes the form

$$
X = \int \left\{\frac{1}{2}\rho_\infty\left(\frac{\partial\varphi}{\partial x_i}\right)^2 + \frac{4p_\infty\rho^2}{\rho_\infty(\rho_\infty + \rho)} + \frac{2a\rho_\infty\rho^2}{\rho_\infty + \rho}\right\} d\mathbf{x}. \tag{11.4}
$$

By virtue of (10.1) and (10.2) there follows immediately from the relation (11.4) the inequality (11.3). Now let $\gamma = 1 + \alpha$, $\alpha > 0$. In the limit as $\alpha \to \infty$ the functional

$$X = \int \left\{ \frac{1}{2}\rho_\infty \left(\frac{\partial \varphi}{\partial x_i}\right)^2 + \frac{2p_\infty \rho^2}{\alpha \rho_\infty(\rho_\infty + \rho)} + \frac{\alpha \rho^2}{\alpha(\rho_\infty + \rho)} \right\} d\mathbf{x}. \tag{11.5}$$

From (11.5) with account for the relations (10.1), (10.2) there again follows the inequality (11.3). Finally, from (11.2), (11.3) there follows the estimate

$$dW_1/dt > -2E_2 + 3K_2 > -2E_2 \quad \text{for } 1 < \gamma \leq 2. \tag{11.6}$$

It is further assumed that the Lagrange theorem inversion condition is satisfied, i.e., the potential energy functional $\Pi_2$ (10.7) on the examined states of rest (10.2) does not have a minimum. This means that among the arbitrarily small (and among the finite) disturbances there will be those for which $\Pi_2 = \Pi_{20} < 0$. As the initial values for $t = 0$ we take the disturbance with $K_2(0) = 0$, $\Pi_2(0) = \Pi_{20}$. Then by virtue of (10.7) there holds the inequality $E_2(0) = \Pi_{20} < 0$. In this case there follows from the estimate (11.6) the inequality $W_1 > 2 \mid \Pi_{20} \mid t$, which, with account for the definition of the functional $W_1$ (11.1) reduces to the form

$$I = \int [\rho^2 + \varphi^2]\, d\mathbf{x} > 4|\Pi_{20}|t. \tag{11.7}$$

If we use the integral I (11.7) as a measure of the deviation of the flow from the state of rest, then there is an instability of the following type: for any number $\varepsilon > 0$ the inequality $I < \varepsilon$ is violated after a finite time, no matter how small the amplitude I(0) of the disturbances that are selected as the initial values. This instability is "stronger" than that understood in the conventional Lyapunov definition [1-3], where for the verification of instability the existence of at least one value of $\varepsilon$ is sufficient, and the violation of the condition $I < E$ is examined over an infinite interval of time. We must emphasize that the estimate (11.7) is indicative of an actual physical instability of the states of equilibrium (rest) (2.6), since the ambiguity in the selection of the velocity potential $\varphi(\mathbf{x}, t)$ is removed with the aid of the connection (10.3).

It is also important to note that the heuristic basis for selecting the Lyapunov functional $W_1$ (11.1) was the Hamiltonian formulation of the theorem on the instability of finite-dimensional mechanical systems [2] and the well-known technique of the introduction of the canonical variables in hydrodynamics [25, 26]. Moreover, in the linear approximation $W_1$ (11.1) coincides with the functional W from (6.2). Proof of this fact is obtained immediately after the application of the Gauss theorem [27].

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## REFERENCES


1. A. M. Lyapunov, Stability of Motion [in Russian], GITTL, Moscow–Leningrad (1950).
2. N. G. Chetaev, Stability of Motion. Studies in Analytic Mechanics [in Russian], Izv. Akad. Nauk SSSR, Moscow (1962).
3. N. N. Moiseev and V. V. Rumyantsev, Dynamics of a Body with Fluid-Containing Cavities [in Russian], Nauka, Moscow (1965).
4. V. V. Rumyantsev, "On the theory of motion of rigid bodies with fluid-filled cavities," PMM, **30**, No. 1 (1966).
5. S. Ya. Belov and V. A. Vladimirov, "An example of Lagrange theorem inversion in two-layer fluid hydrodynamics," in: Dynamics of Continuua: Collection of Scientific Works [in Russian], AS USSR, Siberian Division, Institute of Hydrodynamics, No. 84 (1988).
6. V. A. Vladimirov, "On the instability of fluid equilibrium," PMTF, No. 2 (1989).
7. V. A. Vladimirov and V. V. Rumyantsev, "On inversion of the Lagrange theorem for a rigid body with a cavity containing an ideal fluid," PMM, **53**, No. 4 (1989).
8. V. A. Vladimirov and V. V. Rumyantsev, "On inversion of the Lagrange theorem for a rigid body with a cavity containing a viscous fluid," PMM, **54**, No. 2 (1990).
9. Yu. G. Gubarev, "On inversion of the Lagrange theorem in magnetohydrodynamics," PMM, **54**, No. 6 (1990).
10. K. I. Il'in, "On the stability of a baroclinic vortex," Izv. AN SSSR. FAO, **27**, No. 5 (1991).

11. S. Chandrasekhar, Ellipsoidal Figures of Equilibrium [Russian translation], Mir, Moscow (1973).
12. J. H. Jeans, "The stability of a spherical nebula," Phil. Trans. Roy. Soc. (London), Vol. 199 (1902).
13. J. H. Jeans, Astronomy and Cosmogony [Russian translation], Cambridge University Press, Cambridge (1928).
14. S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability [Russian translation], Clarendon Press, Oxford (1961).
15. Z. S. Quian, E. A. Spiegel, and M. R. E. Proctor, "The gravitational instability of a gaseous slab," Stability and Appl. Anal. of Continuous Media, **1**, No. 1 (1991).
16. S. Chandrasekhar, "The gravitational instability of an infinite homogeneous turbulent medium," Proc. Roy. Soc. (London), **210A**, (1951).
17. S. Chandrasekhar and E. Fermi, "Problems of gravitational stability in the presence of a magnetic field," Astrophys. J., **118** (1953).
18. S. Chandrasekhar, "The gravitational instability of an infinite homogeneous medium when Coriolis force is acting and a magnetic field is present," Astrophys. J., **119**, (1954).
19. W. Fricke, "On the gravitational stability in a rotating isothermal medium," Astrophys. J., **120**, (1954).
20. H. L. Helfer, "Waves of finite amplitude in an infinite homogeneous medium," Astrophys. J., **115**, (1954).
21. J.-L. Tassoul and G. Pellieux, "Nonlinear condensation in self-gravitating media. I. A numerical approach," Astrophys. J., **171**, (1972).
22. J.-L. Tassoul and G. Pellieux, "Nonlinear condensations in self-gravitating media. II. An analytical approach," Ibid.
23. J.-L. Tassoul and H. Dedic, "Finite-amplitude disturbances in self-gravitating mediam" Astron. and Astrophys., **26**, (1973).
24. E. T. P. Liang, "Nonlinear periodic waves in a self-gravitating fluid and galaxy formation," Astrophys. J., **230**, (1979).
25. B. I. Davydov, "The variational principle and canonical equations for an ideal fluid," DAN SSSR, **69**, No. 2 (1949).
26. H. Bateman, Partial Differential Equations of Mathematical Physics [Russian translation], Cambridge Univ. Press, Cambridge (1959).
27. G. M. Fikhtengol'ts, Fundamentals of Mathematical Analysis. Vol. 2 [in Russian], Nauka, Moscow (1968).